

Heterotic type IIA duality with fluxes – towards the complete story

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Abstract

In this paper we study the heterotic type IIA duality when fluxes are turned on. We show that many of the known fluxes are dual to each other and claim that certain fluxes on the heterotic side require that the type IIA picture is lifted to M or even F-theory compactifications with geometric fluxes.

1 Introduction

Flux compactification has been a central research topic in the past ten years. One of the main interests comes from the fact that fluxes generate potentials for certain moduli fields which can be fixed in this way. Type II theories have been studied extensively and there are examples when all moduli have been fixed by fluxes. This is however a rather ungeneric situation and most of the time there are moduli fields which do not appear in the flux generated superpotential. The number and types of moduli fields which appear/do not appear in the superpotential largely depends on the scheme of compactification considered. It is therefore important to know all the possible fluxes which can be turned on in a given setup. A powerful way to control the fluxes is by using dualities. There exist several instances when new fluxes have been discovered in this way. It was first realised in [1] that T and S-dualities trade flux type IIB backgrounds for torsional heterotic geometries.¹ Other examples include mirror symmetry where the NS-NS flux is mapped to compactifications on manifolds with $SU(3)$ structure [4, 5]. The same was noticed in toroidal compactifications where again the NS-NS flux was mapped by T-duality into compactifications on twisted tori [6]. More complete pictures of these dualities in the presence of fluxes have appeared once manifolds with $SU(3) \times SU(3)$ structure [7]-[12] T-folds [13]-[18] or the notion of non-geometric [19] fluxes have been introduced. A systematic way of finding all possible fluxes in a given setup is to start from a certain background with fluxes and apply all possible duality transformations (like S and T-dualities, $SL(2, \mathbf{Z})$ or even exceptional duality groups) until no new fluxes are generated. In this way, many new fluxes have been discovered in [20]-[26].

One very interesting case, which we will study in this paper, is the heterotic type IIA duality with fluxes. The interest comes mainly from the fact that the two compactifications schemes are rather different in nature. This duality in the presence of fluxes was first analysed in [27] where only a small part of the fluxes which can be turned on in the heterotic picture were mapped the type IIA side. This was further extended in [28] where it was realised that fluxes on the heterotic side correspond to geometric fluxes (manifolds with $SU(3)$ structure) on the type IIA side. Finally, in [29] it was argued that certain fluxes in the heterotic picture do not find a correspondent in type IIA compactifications, but one has to lift this to M-theory compactifications on seven-dimensional manifolds with $SU(3)$ structure.

In this paper we continue the study of heterotic type IIA duality with fluxes. We first give an extended version of the results in [29]. Considering fluxes which gauge isometries in the vector multiplet sector, we show that the whole amount of fluxes which appear in the M-theory picture can be understood on the other side by heterotic compactifications with duality twists as spelled out in [14, 30]. We further extend this analysis and conjecture that “R-fluxes” in the heterotic picture correspond to F-theory compactifications on eight-dimensional manifolds with $SU(3)$ structure.

We make a similar analysis for the fluxes which gauge isometries in the hyper-multiplet sector. Even though in this case there are more fluxes on the type IIA side which gauge isometries in the hyper-multiplet sector, we still find that for certain setups on the heterotic side one needs to consider M and even F-theory compactifications.² Therefore the

¹For recent similar examples where non-geometric heterotic solutions are dualised to geometric F-theory backgrounds see [2, 3].

²M-theory compactifications on twisted seven-dimensional manifolds of the type we consider here were

broadest picture we shall obtain will be that F-theory compactifications on eight-manifolds which are obtained by fibering a Calabi–Yau over a torus in a particular way are dual to heterotic string compactifications on six manifolds with $SU(2)$ structure. However even in this case there will still be fluxes on either side which are unaccounted for. It is possible that most of this mismatch comes from the fact that on the heterotic side we ignore a large part of the hyper multiplet space, namely that related to the gauge bundle deformations as this space can not be treated in a generic fashion. Hopefully for a well defined setup one can account for all the fluxes which can be turned on.

The outline of the paper is the following. In section 2, we start by recalling few facts about compactifications of type IIA and heterotic strings to four dimensions with $N = 2$ supersymmetries. We do this mostly for fixing the conventions and notations. We also explain briefly the effect of turning on fluxes in order to motivate our analysis. In section 3, we analyse fluxes on both heterotic and type IIA side which gauge isometries in the vector multiplet sector. We first recall M-theory compactifications on seven dimensional manifolds with $SU(3)$ structure which lead to gaugings in the vector multiplet sector. Then we show that all these fluxes can be found in heterotic compactifications with duality twists. Then we conjecture that R-fluxes in heterotic compactifications can be accounted for in a F-theory setup. We also make some speculations about how one can obtain a charged dilaton in the heterotic picture. In section 4 we turn to those fluxes which gauge isometries in the hyper-multiplet sector. After we review the known facts about the duality when ordinary fluxes are turned on we recall the results of heterotic compactifications on manifolds with $SU(2)$ structure and show that part of the content of the theory can again be obtained on the type IIA side if one considers M-theory compactifications on manifolds with $SU(3)$ structure. To recover the full picture we conjecture again that one needs to go to F-theory compactifications. In the appendices we gathered the most important information about $N = 2$ gauged supergravity in four dimensions and about the structure of the vector multiplet sector in heterotic string compactifications. We also derive here the gaugings which appear in the vector multiplet sector in the compactifications with duality twists.

2 Preliminaries

As explained in the Introduction, we are interested in the duality between heterotic and type IIA string compactified to four dimensions with $N = 2$ supersymmetry. On the heterotic side such situations can be obtained upon compactifications on $K3 \times T^2$ manifolds, while in type IIA we deal with $K3$ fibered Calabi–Yau manifolds [32]. $N = 2$ supersymmetry ensures that the scalar sector of the theory splits into a part describing the vector multiplets and one which describes the hyper-multiplets. In the presence of fluxes certain isometries of the scalar manifold are gauged and this leads to a mixing between these two sectors. However it still makes sense to distinguish between them and moreover to distinguish between fluxes which gauge isometries in the vector and those which gauge isometries in the hyper-multiplet sector. In the following we shall review these compactifications without fluxes and in the end explain briefly the effect of turning on fluxes.

recently studied in [31].

2.1 Heterotic on $K3 \times T^2$

Let us start with the heterotic side. As explained above we consider the heterotic string compactified on $K3 \times T^2$. For simplicity we shall not take into account the non-Abelian structure of the heterotic supergravity in ten dimensions and restrict ourselves to the Cartan algebra of the ten-dimensional gauge group. As the aim of the paper is to make general remarks about the heterotic–type IIA duality with fluxes, these aspects are not going to be important for us. We nevertheless have to keep in mind that we have to satisfy the heterotic Bianchi identity and turn on a certain gauge bundle on $K3$ which actually breaks the original gauge group. Therefore the precise resulting gauge group in four dimensions is not known unless one specifies explicitly the gauge bundle which is turned on in the background and we shall work with a generic $U(1)^{n_v+1}$ for some number n_v of vector multiplets which depends on the details of the compactification and which we shall leave arbitrary. Part of the gauge fields come from the Cartan sub-algebra of the surviving gauge group while four of the gauge fields emerge from the Kaluza-Klein vectors on the torus as well as form the B -fields with one leg on the torus.³ One of these gauge fields (or a combination thereof) is the graviphoton and is part of the gravity multiplet, while the remaining gauge fields are part of vector multiplets. In four dimensions, each vector multiplet also contains one complex scalar, and in the present setup $2 \cdot (n_v - 3)$ of them come from the ten-dimensional gauge fields on the torus, four from the T^2 moduli and from the B -field with both legs on the torus, and the last two from the ten-dimensional dilaton and from the universal axion, the Poincare dual of the four-dimensional B -field. All these fields span the homogeneous space

$$\mathcal{M}_v = \frac{SU(1,1)}{U(1)} \times \frac{SO(2, n_v - 1)}{SO(2) \times SO(n_v - 1)} . \quad (2.1)$$

Beside the vector multiplets there is a certain number of hyper-multiplets. Among the scalars in the hyper-multiplets there are the $K3$ moduli together with the B -field on $K3$ and the $K3$ volume which span the homogeneous space

$$\mathcal{M}_h = \frac{SO(4, 20)}{SO(4) \times SO(20)} \quad (2.2)$$

In the hyper-multiplets there are also the scalar fields which parameterise the deformations of the gauge bundle. The number of these fields and the geometry of the full space spanned by these scalars highly depends on the background gauge bundle and therefore in the following we shall only concentrate on the part which is due to the $K3$ moduli.

The four-dimensional theory obtained from this compactification is a $N = 2$ supergravity coupled to the vector multiplets and hyper-multiplets described above. The bosonic part of the action is given by the general formula given in the appendix, (A.5). In this case $x^i = (s, t, u, n^a), i = 1, \dots, n_v; a = 4, \dots, n_v$ denote collectively the complex scalars in the vector multiplets, which in terms of the compactification fields are given in (B.2)

³In general it may happen that some of these gauge fields are also broken in the compactification due to the particular gauge bundle which is chosen on $K3$. These gauge fields are particularly important in the gaugings which occur in the vector multiplet sector and we shall assume that they survive the compactification to four dimensions. In the absence of (some of) these gauge fields, many – and probably, the most interesting – of the vector multiplet gaugings will be absent.

and $F^I = dA^I$ denote the Abelian field strengths of the vector fields A^I , $I = 0, \dots, n_v$. Moreover, $g_{i\bar{j}}$ is the corresponding Kähler metric derived from the Kähler potential

$$K = -\ln i(\bar{s} - s) - \ln \frac{1}{4}[(t - \bar{t})(u - \bar{u}) - (n^a - \bar{n}^a)(n^a - \bar{n}^a)] . \quad (2.3)$$

and the gauge coupling matrix is given by (B.5). All the information related to the vector multiplet sector is encoded in (B.4), but due to the absence of the prepotential the derivation of the gauge coupling matrix has to follow the general line developed in [33] rather than using the formula (A.4)

Finally, the scalars in the hyper-multiplets q^u and the corresponding quaternionic metric h^{uv} can not be precisely written down in general and therefore we shall mostly concentrate on the scalars coming from the $K3$ moduli. Let us briefly recall how these fields appear. $K3$ is a hyper-Kähler manifold and thus has a triplet of complex structures J^x , $x = 1, 2, 3$. The metric deformations on $K3$ can be parameterised by deformations of these complex structures. Let us expand the complex structures J^x in a basis of harmonic forms on $K3$, ω^A , $A = 1, \dots, 22$

$$J^x = \zeta_A^x \omega^A . \quad (2.4)$$

The parameters of these expansions, ζ_A^x , will appear in the effective four-dimensional theory as scalar fields. However, not all of them represent independent degrees of freedom but their variations are subject to the constraints

$$\zeta_A^x \delta \zeta^{yA} = 0 , \quad (2.5)$$

which leave us with 57 possible deformations. To these we add the volume modulus as well as the massless modes which come from the B -field which we parameterise as

$$B = b_A \omega^A . \quad (2.6)$$

Altogether these fields span the homogeneous space $SO(4, 20)/SO(4) \times SO(20)$.

One important thing to keep in mind from this section is that for the heterotic compactification the vector multiplet sector is governed entirely by the T^2 part of the compactification while the hyper-multiplet sector comes from the $K3$ part. Therefore, in this case, the split between vector and hyper-multiplets comes in naturally due to the split of the compactification manifold. We shall see shortly that the same sort of split can be observed also for the fluxes, as fluxes in the T^2 have as effect gaugings of the vector-multiplet isometries, while $K3$ fluxes gauge isometries in the hyper-multiplet sector.

2.2 Type IIA on Calabi–Yau three-folds

Let us now discuss the compactification of type IIA supergravity on Calabi–Yau manifolds. For the moment we shall present the general features, and later we shall specialise to the case of $K3$ fibered manifolds which are relevant in the heterotic–type IIA duality. We keep the discussion short and for more details we refer the reader to [34, 35] which we closely follow. In type IIA compactifications on Calabi–Yau manifolds there is also a natural splitting between the vector and hyper-multiplets. This however comes from the Calabi–Yau moduli space property that it splits into a product of the space of Kähler class deformations and the space of complex structure deformations. The first gives the scalars in the vector multiplets while the latter together with the RR axions as well as the

dilaton and four-dimensional axion give the scalars in the hyper-multiplets. Altogether the effective theory in four dimensions is $N = 2$ supergravity coupled to $n_v = h^{1,1}$ vector multiplets and $n_h = h^{2,1} + 1$ hyper-multiplets, where $h^{1,1}$ and $h^{2,1}$ denote the dimensions of the corresponding cohomology groups of the Calabi–Yau manifold under consideration.

Let us denote by ω_i and $\tilde{\omega}^i$, $i = 1, \dots, h^{1,1}$ the harmonic $(1, 1)$ and $(2, 2)$ forms and by (α_A, β^A) , $A = 0, 1, \dots, h^{2,1}$ a real basis for the harmonic 3-forms on the Calabi–Yau manifold. These forms are taken to satisfy

$$\int_X \omega_i \wedge \tilde{\omega}^j = \delta_i^j, \quad \int_X \alpha_A \wedge \beta^B = \delta_A^B, \quad (2.7)$$

with all other integrals vanishing.

In order to obtain the low energy degrees of freedom we expand the ten-dimensional form fields in the above harmonic forms. From the expansion of the 3-form potential C_3 we obtain $h^{1,1}$ gauge fields, A^i , and $2(h^{2,1} + 1)$ scalars (RR-axions), ξ^A and $\tilde{\xi}_A$

$$C_3 = A^i \omega_i + \xi^A \alpha_A - \tilde{\xi}_A \beta^A. \quad (2.8)$$

The remaining fields come from the metric deformations of the Calabi–Yau manifold. Let us denote by x^i the complexified Kähler deformations which are given by

$$J + iB = x^i \omega_i, \quad (2.9)$$

where J denotes the Kähler form on the Calabi–Yau manifold and B is the B -field on the internal space. We can introduce projective coordinates in the form

$$x^i = \frac{X^i}{X^0}, \quad (2.10)$$

and it turns out that the special Kähler geometry on this space is given by the prepotential

$$\mathcal{F} = -\frac{1}{6} \frac{\mathcal{K}_{ijk} X^i X^j X^k}{X^0}, \quad (2.11)$$

where \mathcal{K}_{ijk} are the triple intersection numbers on the Calabi–Yau manifold

$$\mathcal{K}_{ijk} = \int_{Y_6} \omega_i \wedge \omega_j \wedge \omega_k. \quad (2.12)$$

Finally, for the vector multiplet sector, the gauge coupling matrix \mathcal{N}_{IJ} is given by the general $N = 2$ formula (A.4) for the prepotential (2.11).

The complex structure deformations can be obtained from the expansion of the holomorphic $(3, 0)$ form on the Calabi–Yau manifold in the (real) basis of three-forms (α_A, β^A)

$$\Omega = Z^A \alpha_A - \mathcal{G}_A \beta^A, \quad (2.13)$$

where \mathcal{G}_A are the derivatives of the prepotential corresponding to the special Kähler geometry which describes the complex structure moduli space and Z^A are projective coordinates on this space.

With the above notations the action for the effective theory obtained by compactifying the type IIA string on Calabi–Yau manifolds can be put in the form (A.5) where the hyper-scalars q^u denote collectively $q^u = (z^a, \phi, a, \xi^A, \tilde{\xi}_A)$, with z^a the complex structure deformations given by

$$z^a = \frac{Z^a}{Z^0}, \quad a = 1, \dots, h^{2,1}, \quad (2.14)$$

ϕ the dilaton, a the axion which is Poincare dual to the four-dimensional B -field, and ξ^A and $\tilde{\xi}_A$ defined in (2.8).

2.3 Turning on fluxes

Let us explain Briefly the effect of turning on fluxes in the above compactifications. We first concentrate on the fluxes which can be turned on in the heterotic picture. The p-forms available for turning on fluxes are the gauge field strengths two forms F^I and the three-form H , the field strength of the NS-NS B -field. Inside $K3$ there are 22 two-cycles on which we can turn on the fluxes F^I and these lead to gaugings in the hyper-multiplet sector. There is also the possibility that we turn on the the fluxes F^I along T^2 and this turns out to gauge isometries in the vector multiplet sector. Finally H can only be turned on with one leg on the torus and two legs along some $K3$ two-cycle. Since the B -field with one leg on the torus is again one of the four-dimensional gauge fields we can think of the H fluxes again as fluxes for these gauge fields on $K3$. Therefore, we shall only distinguish between fluxes strictly inside $K3$ and fluxes along T^2 . The first gauge isometries in the hyper-multiplet space while the latter gauge isometries in the vector multiplet space.

Other types of fluxes on the heterotic side can be obtained by deforming the compactification manifold. We can consider compactifications with duality twists – also known as T-folds. Such compactifications, when applied to our case lead to gaugings in the vector multiplet sector as we shall explain in section 3. Other deformations include manifolds with $SU(2)$ structure and it turns out that these lead to gaugings in the hyper-multiplet sector.

On the type IIA side there are several fluxes available. First of all there is the three-form flux for H , the field strength of the NS-NS antisymmetric tensor B . Moreover, we can turn on RR fluxes which comprise all even forms. All these fluxes gauge isometries in the hyper-multiplet sector. There are also generalisations of these fluxes which include geometric fluxes (manifolds with $SU(3)$ structure) or non-geometric fluxes (manifolds with $SU(3) \times SU(3)$ structure). We shall discuss all these fluxes in more detail in section 4, but for the moment it is important to note that all these fluxes only gauge isometries in the hyper-multiplet sector.

The purpose of the rest of the paper is to try to match the various fluxes discussed above between the heterotic and type IIA picture. As we have already pointed out there are no fluxes strictly within type IIA theory which can lead to gaugings in the vector multiplet sector. It was proposed that the heterotic fluxes which gauge isometries in the vector multiplet space can actually be described in M-theory rather than in type IIA. We shall review this proposal in the next section and present arguments that further extensions of this proposal involve also F-theory.

3 Vector multiplet gaugings

From the brief review in the previous section we have seen that in the case of the heterotic string compactifications we can easily obtain gaugings in both hyper- and vector multiplet sector. On the other hand, in the type IIA picture there are no gaugings in the vector multiplet sector. In this section we shall concentrate on the vector multiplet sector and explain how we can obtain gaugings in this sector in the context of type IIA theories.

3.1 Heterotic compactifications with gauge field fluxes on T^2

At the beginning of this section let us briefly recall the effect of fluxes which we can turn on on T^2 in heterotic string compactifications. Let us consider fluxes of the type

$$\int_{T^2} F^a = f^a, \quad a = 4, \dots, n_v. \quad (3.1)$$

Such compactifications were considered in [36, 37] and here we only briefly recall the results. Later on we will also present a more detailed calculation from where these results can be obtained. We will only be interested in the vector multiplet sector whose structure is explained in appendix B.

The result is that some of the scalars in the vector multiplets become charged and their covariant derivatives read

$$\begin{aligned} Dt &= \partial t - \sqrt{2} n^a f^a A^1 + f^a A^a, \\ Dn^a &= \partial n^a - \frac{1}{\sqrt{2}} f^a (A^0 + u A^1), \end{aligned} \quad (3.2)$$

Moreover the gauge group becomes non-Abelian and the field strengths for the gauge fields are given by

$$\begin{aligned} F^0 &= dA^0, \\ F^1 &= dA^1, \\ F^2 &= dA^2 + f^a A^a \wedge A^1, \\ F^3 &= dA^3 - f^a A^a \wedge A^0, \\ F^a &= dA^a - f^a A^0 \wedge A^1. \end{aligned} \quad (3.3)$$

There is also a potential which is generated, but it is completely fixed by the $N = 2$ supersymmetry from the above data and hence we shall not be concerned with it in the following.

3.2 M-theory compactifications on seven-dimensional manifolds with $SU(3)$ structure

We shall now explain what is the correspondent in the type IIA setup of the picture presented above. As it was shown in [29], we are led to consider M-theory compactifications on seven-dimensional manifolds. We review the results in [29, 31] below. The main insight for the origin of the fluxes which produce gaugings in the vector multiplet sector comes from studying the heterotic type IIA duality with heterotic fluxes on T^2 from the perspective of the five-dimensional duality between heterotic string compactified on $K3 \times S^1$ and M-theory compactified on Calabi–Yau manifolds. Upon further compactifying on a circle we end up with the heterotic type IIA duality in four dimensions that we are interested in. From this point of view, the heterotic fluxes appear only in this last step. These fluxes can be thought of as monodromies of the scalars in the 5d vector multiplets around the circle which takes us down to four dimensions. We can try to do something similar in the M-theory case in the S^1 compactification. The 5d vector multiplet scalars come from the Kähler moduli of the Calabi–Yau manifold and the monodromies around the circle

imply that actually the Calabi–Yau manifold is fibered over the circle. Denoting again the harmonic two-forms on the Calabi–Yau by ω_i we describe the monodromy by

$$d\omega_i = M_i^j \omega_j \wedge dz \quad (3.4)$$

where the constants M_i^j form the twist (monodromy) matrix while dz describes the circle direction.⁴ The compactification on such manifolds was proposed and carried out in [29] and in the following we briefly summarise the results. The action in four dimensions is again given by (A.5) but now with covariant derivatives replacing the ordinary derivatives on the scalar fields in the vector multiplets

$$Dx^i = dx^i - k_I^i A^I, \quad \text{with} \quad k_0^j = -x^k M_k^j, \quad k_i^j = M_i^j. \quad (3.5)$$

Moreover, the field strengths of the four-dimensional gauge fields are modified as

$$F^I = dA^I + \frac{1}{2} f_{JK}^I A^J \wedge A^K, \quad \text{with} \quad f_{ij}^0 = 0 = f_{ij}^k, \quad f_{i0}^j = -M_i^j, \quad (3.6)$$

where $i, j = 1, \dots, n_v$ and $I, J = 0, 1, \dots, n_v$. Finally the action has to be supplemented by the Chern-Simons generalised term

$$S_{gCS} = -\frac{1}{6} \int_{M_4} M_i^l \mathcal{K}_{jkl} A^i \wedge A^j \wedge dA^k,$$

which appears in addition to the standard action (A.5) because of the lack of invariance of the prepotential under the gauge transformations [29].

In the above setup, the parameters M_i^j are subject to the constraint

$$M_i^l \mathcal{K}_{jkl} + M_j^l \mathcal{K}_{kil} + M_k^l \mathcal{K}_{ijl} = 0, \quad (3.7)$$

which comes from the fact that the volume of the Calabi–Yau manifold (which in five dimensions is a member of a hyper-multiplet) should not change as we move along the circle.

So far the discussion was general and can apply in principle for any Calabi–Yau manifold. The constraint (3.7) on the other hand tells us that the moduli space of Kähler deformations admits an isometry which is not a generic property of Calabi–Yau manifolds. For the heterotic–type IIA duality, the relevant Calabi–Yau manifolds are $K3$ fibrations over a \mathbf{P}_1 base [32] and the intersection numbers have the following structure

$$\mathcal{K}_{123} = -1, \quad \mathcal{K}_{1ab} = 2\delta_{ab}, \quad a, b = 4, \dots, h^{(1,1)} = n_v, \quad (3.8)$$

where the index 1 denotes the base and the indices 2 and 3 denote other two-cycles which are singled out. In such a case the solution to the constraint (3.7) can be parameterised as

$$m_2 \equiv M_2^2, \quad m_a \equiv M_a^2, \quad m_3 \equiv M_3^3, \quad \tilde{m}_a \equiv M_a^3, \quad m_b^a \equiv -M_a^b, \quad (3.9)$$

where $m_b^a = -m_a^b$ and the other matrix elements are then given by

$$\begin{aligned} M_2^a &= \frac{1}{2} \tilde{m}_a, & M_3^a &= \frac{1}{2} m_a, & M_a^a &= -\frac{1}{2} M_1^1 = \frac{1}{2} (m_2 + m_3), \\ M_1^{2,3} &= M_1^a = M_a^1 = M_{2,3}^1 = M_2^3 = M_3^2 = 0. \end{aligned} \quad (3.10)$$

⁴Note that we are indeed dealing with a $SU(3)$ structure as manifolds with $SU(3)$ holonomy in seven dimensions necessarily have $M_i^j = 0$ in the equation above.

One of the main task of this section is to find the heterotic correspondent of all the parameters above. In [29] it was shown that the parameters \tilde{m}_a correspond to gauge field fluxes on the heterotic side. The case $m_2 + m_3 \neq 0$ is a bit more subtle and we shall discuss some ideas at the end of this section. So, for the moment we consider that $m_2 + m_3 = 0$ and show that all the parameters above can be recovered in the compactification of the heterotic supergravity with duality twists.

In order to be able to compare the type IIA (M-theory) picture with the heterotic one we should first perform an electric-magnetic duality in order to be in the same symplectic frame on both sides. This requires to exchange one of the gauge fields (A_1 in the case at hand) with its magnetic dual. For the case $m_2 = -m_3 = m$ one can immediately see from (3.5) that no scalar fields are charged under A^1 and from (3.6) that the field strength for this vector field is simply $F^1 = dA^1$. The only place where this gauge field appears non-trivially is in the generalised Chern-Simons term (3.7). Let us write this term in more detail for the specific parameters from (3.9).

$$S_{gCS} = \frac{1}{3} \int (mA^2 \wedge A^3 - \tilde{m}_a A^2 \wedge A^a - m_a A^3 \wedge A^a - m_b^a A^b \wedge A^a) \wedge dA^1 \quad (3.11)$$

$$- \frac{1}{6} \int d(mA^2 \wedge A^3 - \tilde{m}_a A^2 \wedge A^a - m_a A^3 \wedge A^a - m_b^a A^b \wedge A^a) \wedge A^1$$

Integrating by parts in the last term we end up with

$$S_{gCS} = \frac{1}{2} \int (mA^2 \wedge A^3 - \tilde{m}_a A^2 \wedge A^a - m_a A^3 \wedge A^a - m_b^a A^b \wedge A^a) \wedge dA^1. \quad (3.12)$$

We see that the field A^1 actually appears in the action only through its field strength $F^1 = dA^1$ and hence, can be easily dualised. The result of the dualisation is that the generalised Chern-Simons term disappears while the field strength of the magnetic dual gauge field \tilde{A}^1 has the form

$$G^1 = d\tilde{A}^1 - mA^2 \wedge A^3 + \tilde{m}_a A^2 \wedge A^a + m_a A^3 \wedge A^a + m_b^a A^b \wedge A^a. \quad (3.13)$$

The remaining field strengths have the form

$$\begin{aligned} F^0 &= dA^0; \\ F^2 &= dA^2 + mA^0 \wedge A^2 + m_a A^0 \wedge A^a \\ F^3 &= dA^3 - mA^0 \wedge A^3 + \tilde{m}_a A^0 \wedge A^a \\ F^a &= dA^a + \frac{1}{2} \tilde{m}_a A^0 \wedge A^2 + \frac{1}{2} m_a A^0 \wedge A^3 + m_a^b A^0 \wedge A^b. \end{aligned} \quad (3.14)$$

We can therefore read off the following non-vanishing structure constants of the gauge algebra

$$\begin{aligned} f_{23}^1 &= -m; & f_{2a}^1 &= \tilde{m}_a; & f_{3a}^1 &= m_a; & f_{ab}^1 &= 2m_a^b; & f_{02}^2 &= m; & f_{0a}^2 &= m_a; \\ f_{03}^3 &= -m; & f_{0a}^3 &= \tilde{m}_a; & f_{02}^a &= \frac{1}{2} \tilde{m}_a; & f_{03}^a &= \frac{1}{2} m_a; & f_{0a}^b &= m_a^b. \end{aligned} \quad (3.15)$$

Finally the gauged isometries are the same as in (3.5) and we can write explicitly

$$\begin{aligned} D_\mu x^1 &= \partial_\mu x^1 \\ D_\mu x^2 &= \partial_\mu x^2 + (mx^2 + m_a x^a) A_\mu^0 - mA_\mu^2 - m_a A_\mu^a; \\ D_\mu x^3 &= \partial_\mu x^3 + (-mx^3 + \tilde{m}_a) A_\mu^0 + mA_\mu^3 - \tilde{m}_a A_\mu^a; \\ D_\mu x^a &= \partial_\mu x^a + (\frac{1}{2} \tilde{m}_a x^2 + \frac{1}{2} x^3 - m_a^b x^b) A_\mu^0 - \frac{1}{2} \tilde{m}_a A_\mu^2 - \frac{1}{2} m_a A_\mu^3 + m_a^b A_\mu^b. \end{aligned} \quad (3.16)$$

Reading off the killing vectors from the above one can check explicitly that the relation

$$[k_I, k_J] = f_{IJ}^K k_K , \quad (3.17)$$

holds for the structure constants (3.15).

3.3 Heterotic compactifications on $K3 \times T^2$ with duality twists

In the previous subsection we have reviewed the structure of M-theory compactifications on manifolds with $SU(3)$ structure which produce gaugings in the vector multiplet sector. In the following we will show that the same result can be obtained from heterotic string compactifications with duality twists. The $K3$ part of the compactification will be a “spectator” in the current section and we shall be interested only in the T^2 part. The compactification on $K3$ will be assumed to follow in a straightforward manner.

Heterotic compactifications on tori in the presence of fluxes was initially studied in [36]. Here it was shown that the gauge field fluxes, the H -fluxes and the geometric fluxes coming from the twisting of the compactification torus fit nicely in the $O(d, d+16)$ framework of the compactified theory. More recently this setup was generalised in order to include compactifications with (T-)duality twists which are also termed as non-geometric backgrounds [14, 30]. In this section we shall use these recent results in order to show that one can obtain precisely all the flux parameters which were described in the previous section.

The main idea of the duality twists compactification is that one can split the d -dimensional torus into a product $T^{d-1} \times S^1$. The compactification on the $d-1$ torus gives a theory with a $O(d-1, d+15)$ duality symmetry. This can be further compactified on the last S^1 allowing also the fields to vary according to the $O(d-1, d+15)$ duality symmetry. In the case at hand we are interested in T^2 compactifications and therefore we split it as $S^1 \times S^1$ and perform a duality twist compactification on the second S^1 . We have to keep in mind that the $K3$ part of the compactification generically breaks also some of the Cartan generators of the original gauge group and therefore the duality group may not be the full $O(1, 17)$ group and we shall generically denote it by $O(1, n_v - 2)$, where n_v denotes the number of vector multiplets in the final four-dimensional theory.

The most general twist matrix as spelled out in [14, 30] has the form

$$N_N^P = \begin{pmatrix} f & 0 & M^b \\ 0 & -f & W^b \\ -W_a & -M_a & S_a^b \end{pmatrix} . \quad (3.18)$$

Based on duality arguments, the structure constants of the gauged $N = 4$ supergravity were found to be given in terms of the twist matrix as

$$f_{0N}^P = N_N^P \quad f_{NP}^1 = N_{NP} , \quad N, P = 2, 3, \dots, n_v \quad (3.19)$$

where the indices 0 and 1 denote the directions in the gauge field space given by the Kaluza-Klein vector on S^1 and the B -field with one leg on S^1 respectively. The indices of the twist matrix N are raised and lowered with the $O(1, n_v - 2)$ invariant

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n_v-3} \end{pmatrix} . \quad (3.20)$$

The above structure constants suggest that we should perform the following identifications in order to match the heterotic and M-theory sides

$$f = m, \quad W^a = \frac{1}{\sqrt{2}}m_a, \quad M^a = -\frac{1}{\sqrt{2}}\tilde{m}_a, \quad S_{ab} = m_{ab}. \quad (3.21)$$

Finally the field strengths of the gauge fields can be written explicitly using the structure constants above and we find

$$\begin{aligned} F^0 &= dA^0 \\ F^1 &= dA^1 + fA^2 \wedge A^3 + M^a A^2 \wedge A^a + W^a A^3 \wedge A^a + \frac{1}{2}S_a{}^b A^a \wedge A^b \\ F^2 &= dA^2 + fA^0 \wedge A^2 - W^a A^0 \wedge A^a \\ F^3 &= dA^3 - fA^0 \wedge A^3 - M^a A^0 \wedge A^a \\ F^a &= dA^a + M^a A^0 \wedge A^2 + W^a A^0 \wedge A^3 + S_b{}^a A^0 \wedge A^b \end{aligned} \quad (3.22)$$

Comparing with (3.14) we see that we have to perform the following identifications

$$A_a^0 \leftrightarrow A_h^0, \quad \tilde{A}_a^1 \leftrightarrow A_h^1, \quad A_a^2 \leftrightarrow -A_h^2, \quad A_a^3 \leftrightarrow A_h^3, \quad A_a^a \leftrightarrow \frac{1}{\sqrt{2}}A_h^a, \quad (3.23)$$

where the subscript a and h refer to the type IIA and heterotic pictures respectively.

In order to fully establish the above identifications we should also check the gaugings which are produced in the case of heterotic compactifications and compare them with the ones obtained in M-theory.

The scalars which are obtained from heterotic string compactification on d -dimensional tori are arranged in $SO(d, d+16)$ matrices [38, 36]. For the case at hand, since we split the compactification on T^2 into two circle compactifications, we have to define two such matrices. For the first S^1 compactification – which we denote by the index 9 – we find from (B.1)

$$\mathcal{M}_{NP} = \begin{pmatrix} g_{99}(1 + \frac{1}{2}g_{99}^{-1}A_9^b A_9^b)^2 & -\frac{1}{2}g_{99}^{-1}A_9^b A_9^b & A_9^a(1 + \frac{1}{2}g_{99}^{-1}A_9^b A_9^b) \\ -\frac{1}{2}g_{99}^{-1}A_9^b A_9^b & g_{99}^{-1} & -g_{99}^{-1}A_9^a \\ A_9^c(1 + \frac{1}{2}g_{99}^{-1}A_9^b A_9^b) & -g_{99}^{-1}A_9^c & \delta_{ac} + g_{99}^{-1}A_9^c A_9^a \end{pmatrix} \quad (3.24)$$

Here g_{99} denotes the ten-dimensional metric with both legs on the circle and A_9^a denote the ten-dimensional gauge fields on S^1 . After this first circle compactification there will be two additional gauge fields – which we denote by V^3 and V^4 – which come from the metric and from the B -field respectively. The other gauge fields which descent directly from the ten-dimensional ones we shall denote by V^a . In terms of the ten-dimensional quantities they are defined as

$$V_{\hat{\mu}}^3 = g_{99}^{-1}g_{\hat{\mu}9}, \quad (3.25)$$

$$V_{\hat{\mu}}^4 = B_{\hat{\mu}9} + \frac{1}{2}A_9^a V_{\hat{\mu}}^a, \quad (3.26)$$

$$V_{\hat{\mu}}^a = A_{\hat{\mu}}^a - A_9^a g_{99}^{-1}g_{\hat{\mu}9} = A_{\hat{\mu}}^a - A_9^a V_{\hat{\mu}}^1, \quad (3.27)$$

where $\hat{\mu}$ denotes the space-time index in nine dimensions. We shall collectively denote these vector fields as $V^N, N = 3, 4, \dots, n_V + 1$ where the index N is in the fundamental representation of the isometry group $SO(1, n_v - 2)$. In the second circle compactification – which we denote by the index 8 – there will appear additional scalars coming from the

nine-dimensional vectors, V_8^N , and again from the metric with both legs on the circle, g_{88} . Together with the scalars above they can be assembled into a $SO(2, n_v - 1)$ matrix

$$\tilde{\mathcal{M}}_{IJ} = \begin{pmatrix} g_{88} + \mathcal{M}_{NP} V_8^N V_8^P + g_{88}^{-1} C^2 & g_{88} C & g_{88}^{-1} C L_{NR} V_8^R + \mathcal{M}_{NR} V_8^R \\ g_{88} C & g_{88}^{-1} & -g_{88}^{-1} L_{NR} V_8^R \\ g_{88}^{-1} C L_{PR} V_8^R + \mathcal{M}_{PR} V_8^R & -g_{88}^{-1} L_{PR} V_8^R & \mathcal{M}_{NP} + g_{88}^{-1} L_{NR} L_{PQ} V_8^R V_8^Q \end{pmatrix}, \quad (3.28)$$

where by C we defined $C = \frac{1}{2} V_8^N V_8^P L_{NP}$ and L was defined in (3.20). The isometries which are gauged in the last step of the compactification can be read from the covariant derivatives which are written generically as

$$D_\mu \tilde{\mathcal{M}}_{IJ} = \partial_\mu \tilde{\mathcal{M}}_{IJ} + f_{IK}^L \tilde{\mathcal{M}}_{LJ} A_\mu^K + f_{JK}^L \tilde{\mathcal{M}}_{IL} A_\mu^K, \quad (3.29)$$

where the non-vanishing structure constants are given in (3.19). The above formula is nothing but the standard result for $N = 4$ gauged supergravities. The $K3$ part of the compactification does not modify this structure. At most, some of the gauge fields will be broken and this amounts to cut the corresponding lines and columns from the scalar matrix above but without modifying its structure. Thus, the result above is not so mysterious and only descends from the $N = 4$ theory which appears in tori compactifications. The only non-trivial point so far is the assignment of the structure constants (3.19) which has been found in [30].

The final result is nevertheless a $N = 2$ gauged supergravity and in order to be able to compare to the type IIA/M-theory side we need to rewrite the above results in a $N = 2$ language. Using the definitions of the correct $N = 2$ complex scalar fields in four dimensions in terms of the matrix $\tilde{\mathcal{M}}_{IJ}$ which are given in (B.1) it is just a matter of straightforward algebraic manipulations to derive the form of the covariant derivatives on these $N = 2$ fields. The explicit calculation is done in the appendix and in the following we present the final result. Setting all the parameters in (3.31) to zero and inserting in (3.33) we find

$$\begin{aligned} D_\mu u &= \partial_\mu u + f u A_\mu^0 + \sqrt{2} W^a n^a A_\mu^0 + f A_\mu^2 - W^b A_\mu^b, \\ D_\mu t &= \partial_\mu t - f t A_\mu^0 - \sqrt{2} M^a n^a A_\mu^0 + f A_\mu^3 + M^a A_\mu^a, \\ D_\mu n^a &= \partial_\mu n^a - \frac{M_a}{\sqrt{2}} u A_\mu^0 + \frac{W_a}{\sqrt{2}} t A_\mu^0 - S_a^b n^b A_\mu^0 - \frac{1}{\sqrt{2}} (M_a A_\mu^2 + W_a A_\mu^3 - S_a^b A_\mu^b) \end{aligned} \quad (3.30)$$

It is now clear that identifying the type IIA scalar fields x^1, x^2, x^3 and x^a with the dilaton, u, t and n^a respectively, and using (3.21) and (3.23) that the two low energy action precisely agree.

In [29] it was shown that the heterotic gauge field fluxes on T^2 are mapped into the twist parameters \tilde{m}_a on the M-theory side. Here we have extended that analysis and found the correspondent of the other twist parameters on the heterotic side. We see that as anticipated in [29] the non-geometric fluxes W^a which correspond to the T -dual of the usual fluxes M^a as well as S_{ab} which correspond to twistings of the Cartan torus, find a geometric realisation on the M-theory side as the twist parameters m_a and m_{ab} .

3.4 Generalisations: R-fluxes vs. F-theory

There is a certain generalisation on the heterotic side of the duality which amounts to allow twists which would correspond to T-dualities along non-isometric directions [14, 30]. The

fluxes introduced in this way are known as R-fluxes [19] and describe backgrounds which do not admit geometric interpretations even locally. Formally from the 4d perspective this would mean to introduce another twist matrix \tilde{N} which commutes with the initial one N . Let us parameterise this new twist matrix \tilde{N} like in (3.18)

$$\tilde{N}_N^P = \begin{pmatrix} q & 0 & U^b \\ 0 & -q & V^b \\ -V_a & -U_a & G_a{}^b \end{pmatrix}. \quad (3.31)$$

The additional structure constants which are obtained on top of the ones in (3.19) are

$$f_{1N}^P = \tilde{N}_N^P, \quad f_{NP}^0 = \tilde{N}_{NP}. \quad (3.32)$$

The fact that the matrices N and \tilde{N} commute precisely ensures that the structure constants in (3.19) and (3.32) satisfy the Jacobi identity. Given the structure constants above the $N = 2$ gauged supergravity in four dimensions is in principle fully specified as we have seen in the subsection above. The isometries which are gauged can be derived from the general formula (3.29). Following the calculations in the appendix one finds

$$\begin{aligned} D_\mu u &= \partial_\mu u + (fu + \sqrt{2}W^a n^a)A_\mu^1 + (qu + \sqrt{2}V^a n^a)A_\mu^2 + (f - qn^a n^a - \sqrt{2}U^a n^a u)A_\mu^3 \\ &\quad - (qu^2 + \sqrt{2}V^a n^a u)A_\mu^4 + (V^b n^a n^a - U^b u^2 - \sqrt{2}G_b{}^a n^a u - W^b)A_\mu^b \end{aligned} \quad (3.33)$$

$$\begin{aligned} D_\mu t &= \partial_\mu t - (ft + \sqrt{2}M^a n^a)A_\mu^1 - (qt + \sqrt{2}U^a n^a)A_\mu^2 - (qt^2 + \sqrt{2}U^a n^a t)A_\mu^3 \\ &\quad + (f - qn^a n^a - \sqrt{2}V^a n^a t)A_\mu^4 - (U^b n^a n^a - V^b t^2 + \sqrt{2}G_b{}^a n^a t - M^b)A_\mu^b \end{aligned} \quad (3.34)$$

$$\begin{aligned} D_\mu n^a &= \partial_\mu n^a + \frac{1}{\sqrt{2}}(-M^a u + W^a t - \sqrt{2}S_{ab}n^b)A_\mu^1 + \frac{1}{\sqrt{2}}(-U^a u + V^a t - \sqrt{2}G_{ab}n^b)A_\mu^2 \\ &\quad + \left[-qtn^a - \sqrt{2}U^b n^b n^a - \frac{1}{\sqrt{2}}U^a(ut - n^b n^b) - \frac{1}{\sqrt{2}}M^a \right] A_\mu^3 \\ &\quad + \left[-qun^a - \sqrt{2}V^b n^b n^a - \frac{1}{\sqrt{2}}U^a(ut - n^b n^b) - \frac{1}{\sqrt{2}}W^a \right] A_\mu^4 \\ &\quad + \left[(-U^b u + V^b t - \sqrt{2}G_{bc}n^c)n^a - \frac{1}{\sqrt{2}}G_{ba}(ut - n^c n^c) + \frac{1}{\sqrt{2}}S_{ab} \right] A_\mu^b \end{aligned} \quad (3.35)$$

Furthermore, the potential can be computed from the $N = 2$ formalism. Due to the complicated gaugings above, the form of the potential in terms of the scalar fields u , t and n^a is very involved and we shall not present it here.

Note that the potential can be computed only by relying on the $N = 2$ structure of the resulting theory and can not be derived directly from the compactification. In the heterotic case not even the source of this potential is known and therefore being able to derive the potential by other means may shed light on how to directly compute the potential in such non-geometric compactifications.

Now we would like to ask whether the picture above has any sort of type II dual. As we can see from the gaugings above, the second twist matrix gauges isometries with respect to the gauge field A^1 . In type IIA we saw that such gaugings appear provided the gauge field can be interpreted as the KK vector in a Scherk-Schwarz compactification. Therefore we would need a second circle in the IIA compactification. This naturally makes us consider F-theory compactifications on $CY_3 \times T^2$. Indeed given the heterotic F-theory duality in six dimensions we can further twist the Calabi-Yau manifold over the full T^2 precisely in the same way as we did in the case of M-theory compactifications. Concretely,

denoting the the T^2 directions by $z^{1,2}$ we postulate the following differential relations for the full 8d manifold on which we compactify F-theory

$$d\omega_i = M_i^j \omega_j \wedge dz^1 + \tilde{M}_i^j \omega_j \wedge dz^2 . \quad (3.36)$$

Clearly, in order to assure that the exterior derivative squares to zero the matrices M and \tilde{M} have to commute. Moreover the constraint (3.7) has to be satisfied by both matrices M and \tilde{M} . Parameterising \tilde{M} in the same way as we did for the matrix M , it is tempting to claim that the parameters in \tilde{M} precisely correspond to the parameters from \tilde{N} . In order to show that this picture is precisely dual to the heterotic compactification with R-fluxes we would still need to derive the gaugings and the structure constants of the gauge group on the F-theory side and compare to (3.33) and (3.32). This is not possible in general and one needs to first specify a Calabi–Yau manifold in order to obtain a precise result from F-theory compactifications. Therefore it seems that in general one can not test in detail the above conjecture. There are nevertheless qualitative analyses which point towards the fact that this duality conjecture is right. For example, in six dimensions, after compactifying F-theory on an elliptically fibered Calabi–Yau manifold there will be a number of tensor multiplets which are related to the $(1, 1)$ forms on the base of the fibration. Upon the compactification to four dimensions these forms are supposed to satisfy the relation (3.36) and this will imply that the corresponding tensor fields will pick up a mass in four dimensions. In massless compactifications a tensor multiplet in six dimensions descends to a vector multiplet in four dimensions, but in this case we will be dealing with a vector-tensor multiplet where the tensor field picks up a mass in a Stuckelberg mechanism. On the heterotic side we see no sign of massive tensors as we were able to write the gaugings of the $N = 2$ supergravity only in terms of scalar fields. On the other hand we have to recall that after the compactification we end up in a different symplectic frame, and in order to be able to find agreement between the theories we need to perform an electric-magnetic duality. On the heterotic side this amounts to trade the gauge field A^1 for its magnetic dual. For the gaugings produced by the R-fluxes (3.33) we see that this gauge field appears non-trivially. In order to perform the electric-magnetic duality we need to dualise first the charged scalars to tensor fields with Green-Schwarz couplings. Then, by the electric-magnetic rotation these tensor fields will become massive and it will no longer be possible to dualise them back into scalars. Therefore we see that in the correct symplectic frame we also end up with massive tensors on the heterotic side as it was the case for the F-theory compactification.

Finally, on the F-theory side it may be possible to compute the potential directly from the compactification. The potential will generically contain a piece which has a geometric origin – coming from integrating the Ricci scalar over the eight-dimensional manifold – and a piece due to the four-form flux which is sourced by the non-closure of the $(1, 1)$ forms on the full eight-dimensional space. However it is not clear if the potential can be computed in closed form and we leave this for future research.

3.5 Charged dilaton from fluxes

Before closing this section we wish to make a few comments on a case we have discarded so far. In section 3.2 we have discussed M-theory compactifications on manifolds with $SU(3)$ structure. For the purposes of the duality with heterotic compactifications we have chosen the parameters m_2 and m_3 in (3.9) to sum up to zero. However, strictly from the

M-theory side there is nothing to force such a constraint upon us. After all, a twist which has $m_2 + m_3 \neq 0$ is valid at least in the supergravity limit. From (3.10) we see that the gaugings will be more complicated and the scalar x^1 will also be charged. However, nothing exceptional happens as the gaugings (3.5) were derived in general irrespective of the solution to the constraint (3.7). It is therefore legitimate to ask what would such a choice of parameters correspond to on the heterotic side. In fact we have seen what is the effect in the low energy effective action if these parameters satisfy the relation $m_2 + m_3 = 0$ and therefore we would have to ask what happens in the orthogonal case, ie $m_2 - m_3 = 0$. On the heterotic side, the situation is not so simple. The M-theory field x^1 corresponds to the dilaton on the heterotic side and a charged dilaton is not common at all in heterotic string compactifications. In fact, there are even strong no-go theorems which impose strong constraints on the way the dilaton appears in the compactified theory.

The twist we considered in the M-theory case, $m_2 - m_3 = 0$ has the effect on the heterotic side that it takes the dilaton into minus itself. Such a transformation is not a duality within heterotic string theory but it is a duality of string theory, S-duality, which takes heterotic into type I string. Implementing this duality as an allowed twist in the compactification of the heterotic string may lead to the same outcome as in the original M-theory picture, namely a dilaton which is charged under the four-dimensional gauge group. Now the main question that has to be answered is how this can be reconciled with the no-go theorems mentioned above. These no-go theorems are based on the holomorphy arguments and on the axionic shift symmetry of the scalar super-partner of the dilaton. This axionic field is in fact the Poincare dual of the four-dimensional B -field and therefore one expects that this shift symmetry is always present making the argument in favor of the no-go theorem water tight. On the other hand, what can happen in flux compactifications is that the B -field becomes massive and its dualisation to an axion is no longer possible. This is what we expect that happens in this case so that the argument for the no-go theorem is invalidated. This can be seen easily from the M-theory side as we shall explain in the following.

Let us suppose that the parameters m_a , \tilde{m}_a and m_a^b in (3.9) vanish and the only twist parameters which are non-zero are m_2 and m_3 which we choose equal, ie $m_2 = m_3 = \tilde{m}$. Using (3.10) and (3.5) we find the following covariant derivatives for the scalars in the vector multiplets

$$\begin{aligned} Dx^1 &= dx^1 + 2\tilde{m}A^1 \\ Dx^2 &= dx^1 - \tilde{m}A^2 \\ Dx^3 &= dx^1 - \tilde{m}A^3 \\ Dx^a &= dx^1 - \tilde{m}A^a \end{aligned} \tag{3.37}$$

Now recall that in order to obtain the heterotic picture we have to perform an electric-magnetic duality which exchanges the gauge field A^1 with its magnetic dual. Since this gauge field appears explicitly in the covariant derivative of the field x^1 (more precisely only in the real part of the field), in order to perform the electric-magnetic duality one has to promote the real part of the field x^1 to a tensor field which will become massive by a Stueckleberg mechanism. Therefore on the heterotic side the above proposed ‘‘S-fold’’ compactification necessarily gives rise to a massive B-field in four dimensions.

4 Hyper-multiplet gaugings

So far we focused on the duality in the vector multiplet sector. Now we want to address the same question at the level of the hyper-multiplet sector. Therefore we shall be interested in fluxes which produce gaugings in the hyper-multiplet sector. Such fluxes in the type IIA setting are more common than the ones discussed before which produce gaugings in the vector multiplet sector. Still, the lesson from the previous section will be applicable in this case and we shall again be lead to consider M-theory or even F-theory compactifications. We should nevertheless make it clear that the map in the hyper-multiplet sector can not be well defined generically, but one has to specify precisely the two backgrounds on each side of the duality and therefore the arguments will be less accurate than in the previous section.

4.1 Heterotic compactifications with fluxes on K3 and their type IIA dual

In this subsection we review the known facts about the heterotic type IIA duality with fluxes that gauge symmetries in the hyper-multiplet sector. Let us consider for the moment only the effects produced by the ordinary fluxes. In the heterotic case such fluxes are the ones which can be turned on inside $K3$. We denote these fluxes as

$$F^I = m_I^A \omega_A . \quad (4.1)$$

It turns out, that in this case, the directions which are gauged are Peccei-Quinn isometries related to the B -field with both legs on $K3$, or in other words, the scalars which become charged are the ones defined in (2.6). We find the covariant derivatives for these fields to be [37]

$$Db^A = db^A - m_I^A A^I . \quad (4.2)$$

Along with these gaugings a potential is generated which is in agreement with $N = 2$ supergravity as described in appendix A. The action has the same form as in (A.5) with ordinary derivatives replaced by the covariant derivatives listed above and with a potential term added.

Let us now turn to the type IIA side. In this case there are both RR and NS-NS fluxes which lead to gaugings in the hyper-multiplet sector and we parameterise them as

$$H = q_0^A \alpha_A - p_{0A} \beta^A , \quad F_2 = m^i \omega_i , \quad F_4 = e_i \tilde{\omega}^i . \quad (4.3)$$

The effect of the NS-NS fluxes p^A and q_A is to gauge the shift isometries of the RR scalars ξ^A and $\tilde{\xi}_A$ as, [34],

$$D\xi^A = d\xi^A - p^A A^0 , \quad D\tilde{\xi}_A = d\tilde{\xi}_A - q_A A^0 . \quad (4.4)$$

For the RR fluxes, if both e_i and m^i are present, then the B -field is massive in four dimensions. This situation is difficult to obtain in the heterotic picture and therefore we shall not discuss it in the following. Hence we shall suppose that only the fluxes e_i are non-vanishing.⁵ The net effect of turning on such fluxes is the presence in the four-dimensional

⁵Note that choosing the fluxes e_i is just for convenience as they appear on the same footing as the fluxes m^i . The two appear as electric and magnetic charges and one can switch between them by an appropriate electric-magnetic duality.

theory of a Green-Schwarz coupling of the type $e_i F^i \wedge B$, which upon dualisation of the B -field to the universal axion a shows up in the covariant derivative as

$$Da = da + e_i A^i . \quad (4.5)$$

So far we have discussed only ordinary fluxes for the p-form field strengths which we can turn on in both heterotic and type IIA picture. At this stage it is far from obvious how the duality relation is supposed to work. We shall discuss in the following various generalisations which introduce geometric and non-geometric fluxes and make in this way the situation more symmetric between the two sides. Note however, that a subset of the fluxes above should be mapped into one another as observed in [27]. Indeed, since on the heterotic side the $K3$ should be elliptically fibered, the B field through the \mathbf{P}^1 basis of the fibration should correspond to the universal axion, a , on the type IIA side. This should make it clear that the fluxes for the gauge fields through this cycle on the heterotic side, should precisely correspond to the fluxes e_i in (4.5). The rest of the fluxes in (4.1) were then observed to be related to type IIA compactifications on manifolds with $SU(3)$ structure [28].

Let us define the $SU(3)$ structure by deforming the harmonic forms on the Calabi–Yau to obey [9]

$$d\omega_i = q_i^A \alpha_A - p_{iA} \beta^A , \quad d\alpha_A = -p_{iA} \tilde{\omega}^i , \quad d\beta^A = -q_i^A \tilde{\omega}^i . \quad (4.6)$$

It is not difficult to see that the fields which become charged in this case are the RR axions ξ^A and $\tilde{\xi}_A$. Computing dC_3 from the expansion (2.8) and using (4.6) we immediately find that the covariant derivatives for these fields become

$$D_\mu \xi^A = \partial_\mu \xi^A - q_I^A A^I , \quad D_\mu \tilde{\xi}_A = \partial_\mu \tilde{\xi}_A - p_{IA} A^I . \quad (4.7)$$

In the above we have also included the H -fluxes from (4.3) such that all the vector fields in the four-dimensional theory participate in the gauging. This situation resembles very much the one described in (4.2) with the obvious difference that now there are two scalars in each hyper-multiplet which are charged, compared to (4.2) where there is only one charged scalar. Therefore setting half of the deformation parameters to zero, say $q_i^A = 0$, we recover precisely (4.2) as it was shown in [28].

The case $q_i^A \neq 0$ does not seem to have an immediate analogue on the heterotic side. There is a certain relaxation of the problem once we consider manifolds with $SU(2)$ structure as we will show in the next section, but in the most general case there will still be a mismatch of fluxes between heterotic and type IIA sides. We expect that this mismatch comes from the fact that we are not working with the correct quaternionic space on the heterotic side, but only with the sub-part which is spanned by the $K3$ moduli.

4.2 Heterotic compactifications on manifolds with $SU(2)$ structure and their type IIA duals

In this section we shall review the results of [39] where heterotic string compactifications on manifolds with $SU(2)$ structure were analysed.

Let us consider that the $K3$ manifold in the heterotic compactification is non-trivially fibered over the two-torus. In particular we consider that the $K3$ two-forms obey

$$d\omega_A = T_{\alpha A}^B \omega_B \wedge dz^\alpha , \quad (4.8)$$

where T_α , $\alpha = 1, 2$, represent the twist matrices which are antisymmetric and commute with each other. In [39] such manifolds were shown to represent almost the entire class of manifolds with $SU(2)$ structure which have an integrable product structure. The result for such a compactification is a gauged supergravity where all the $K3$ moduli together with the fields b_A are charged under the gauge group

$$\begin{aligned} D_\mu \zeta_A^x &= \partial_\mu \zeta_A^x - T_{\alpha A}{}^B \zeta_B^x A_\mu^\alpha, \\ D_\mu b_A &= \partial_\mu b_A - T_{\alpha A}{}^B b_B A_\mu^\alpha. \end{aligned} \quad (4.9)$$

The vector fields A_μ^α in the above formula represent the Kaluza Klein vectors corresponding to the two circle directions z^α , $\alpha = 1, 2$. As before, this result represents a $N = 2$ gauged supergravity which for consistency needs also a potential term. This was computed in [39] and was shown to precisely agree with the general formula given in the appendix (A.7).

Now let us turn our attention to the type IIA side. We would like to obtain gaugings of the type (4.9) for all the scalars in the hyper-multiplets. We have learned that fluxes together with manifolds with $SU(3)$ structure lead to (constant) gaugings of the shift symmetries of the RR axions. Gaugings like the ones in (4.9) are not so common in type IIA compactifications with fluxes. However, we have encountered a similar example in section 3 and the way to obtain the desired gaugings was to lift the type IIA compactification to M-theory compactifications on seven-dimensional manifolds with $SU(3)$ structure. Then by appropriate twists, gaugings of the type (4.9) can be obtained and the vector field which participates in the gauging is the KK vector on the M-theory circle. In the following we shall consider a similar setup, but now twist the 3-forms around the M-theory circle [31], as the three forms are the ones which mostly govern the hyper-multiplet sector in type IIA compactifications.⁶ Recall that the three-forms on a Calabi–Yau manifold which satisfy (2.7) can be rotated by a symplectic transformation. We shall use this symplectic symmetry in defining the twisting. Consider the following dependence on the M-theory coordinate

$$d \begin{pmatrix} \alpha_A \\ \beta^A \end{pmatrix} = \begin{pmatrix} M_A{}^B & 0 \\ 0 & -M_B{}^A \end{pmatrix} \cdot \begin{pmatrix} \alpha_B \\ \beta^B \end{pmatrix} \wedge dy, \quad (4.10)$$

where the twist matrix is symplectic by construction. Here we denoted by y the circle direction in order to avoid confusion with the complex structure moduli of the Calabi–Yau manifold. In order to match the heterotic side we shall also consider that the matrix $M_A{}^B$ is also antisymmetric. A more general symplectic twist does not seem to have an immediate analogue on the heterotic side, but in order to make a more precise statement one would need to have an explicit map of the hyper-multiplets. While the most general case was discussed in [31], in the following we limit ourselves to the Ansatz (4.10) which can be written explicitly

$$d\alpha_A = M_A{}^B \alpha_B \wedge dy, \quad d\beta^A = -M_B{}^A \beta^B \wedge dy. \quad (4.11)$$

Note that this automatically preserves the orthonormalization of the forms α and β which on the full seven-dimensional manifold reads

$$\int_{7d} \alpha_A \wedge \beta^B \wedge dy = \delta_A^B. \quad (4.12)$$

⁶M-theory compactifications on manifolds with $SU(3)$ structure which give rise to potentials for the hyper-scalars were first studied in [40].

Now we proceed in close analogy to [29] and we shall find it more convenient to work with a basis of forms which does not depend on the additional M-theory coordinate and transfer all this dependence on the moduli fields.

Let us concentrate on the scalars in the hyper-multiplets. These fields come from expanding the holomorphic $(3,0)$ form Ω and the three-form gauge potential C_3 in the basis of three-forms like in (2.13) and (2.8). Gauge invariance requires that the fields Z^A , \mathcal{G}_A , ξ^A and $\tilde{\xi}_A$ transform as

$$\delta Z^A = -M_B^A Z^B \epsilon, \quad \delta \mathcal{G}_A = M_A^B \mathcal{G}_B \epsilon, \quad \delta \xi^A = -M_B^A \xi^B \epsilon, \quad \delta \tilde{\xi}_A = M_A^B \tilde{\xi}_B \epsilon, \quad (4.13)$$

under the change $y \rightarrow y + \epsilon$. Let us make a couple of comments here. First, note that the transformation of \mathcal{G}_A above is required by gauge invariance. On the other hand, \mathcal{G}_A are the derivatives of the prepotential \mathcal{G} with respect to Z^A and therefore one can infer its transformation from the definition of \mathcal{G} . Using the holomorphy of the prepotential \mathcal{G} one immediately finds

$$\delta \mathcal{G}_A = \mathcal{G}_{AB} \delta Z^B = -\mathcal{G}_{AB} M_C^B Z^C \epsilon. \quad (4.14)$$

Comparing with the corresponding transformation from (4.13) we find

$$-\mathcal{G}_{AB} M_C^B = M_A^B \mathcal{G}_{BC} \quad (4.15)$$

This means that for a given prepotential, the possible twists are given by the solutions to the above constraint. This is precisely the analogue of the condition (3.7) found in the previous section when twisting the harmonic $(1,1)$ forms of the Calabi–Yau manifold over the circle. For a generic prepotential we expect no isometry of the special geometry defined by it and therefore no matrix M will satisfy the constraint. Here however we shall consider that there are certain isometries of the special geometry and therefore the constraint will have non-trivial solutions. We do this assumption because ultimately we are interested in mapping this compactification to heterotic strings on $K3 \times T^2$ and we know that the quaternionic space of the hyper-scalars contains the $K3$ moduli space $SO(4,20)/SO(4) \times SO(20)$ which originates from a special geometry of the type $SU(1,1)/U(1) \times SO(2,18)/SO(2) \times SO(18)$.

The second observation we want to make here is that not all the Z^A fields are independent degrees of freedom because they are only projective coordinates on the space of complex structure deformations. In many of the calculations it is convenient to fix the gauge by choosing $Z^0 = 1$. For this to be possible in the present context we need that Z^0 does not transform under $y \rightarrow y + \epsilon$. In the following we shall choose to fix the gauge $Z^0 = 1$ at the expense of setting $M^0_A = M^A_0 = 0$. From the perspective of the duality with the heterotic compactifications this is not so bad as the parameters we want to set to zero lead to gaugings of the scalars in the universal hyper-multiplet which is special anyway and we do not focus on it here.

Consequently they will have covariant derivatives of the form

$$\begin{aligned} D_\mu z^a &= \partial_\mu z^a - M_b^a z^b A^0; \\ D_\mu \xi^a &= \partial_\mu \xi^a - M_b^a \xi^b A^0; \\ D_\mu \tilde{\xi}_a &= \partial_\mu \tilde{\xi}_a + M_a^b \tilde{\xi}_b A^0, \end{aligned} \quad (4.16)$$

where now $a = 1, \dots, n_h$. We see that the result above has precisely the same form as the gauging (4.9). The only difference comes from the fact that in the heterotic case

there were two twist matrices T_α , $\alpha = 1, 2$, while in (4.16) there is only one. A similar problem we have encountered in the previous section where we were trying to match the gaugings in the vector multiplet sector in heterotic and type IIA compactifications. There we have argued that in order to restore the duality one has to go all the way up to F-theory compactifications on eight-dimensional manifolds which are obtained from fibering the Calabi–Yau manifold over a T^2 . If we apply the same logic here we will have to introduce a second twist matrix \tilde{M} which commutes with the matrix M . The result then will precisely match the heterotic side. So also in the hyper-multiplet sector the most general gauging which can be obtained on the heterotic side from compactifications on manifolds with $SU(2)$ structure can be mapped to F-theory compactifications. In this case however, the duality only relates geometric backgrounds and there is no non-geometric aspect involved.

4.3 Turning on multiple fluxes

So far we have only turned on very specific types of fluxes at a time and we did not analyse what happens if we try to turn on more fluxes simultaneously.

First of all let us note that in the last case studied, a vev for the scalars ξ and $\tilde{\xi}_a$ produces a term in the covariant derivatives (4.16) similar to the one in (4.7). Therefore, the effect of the gaugings in (4.7) can be simply removed by shifting the vev for the scalars ξ^A and $\tilde{\xi}_A$ in an appropriate way. In this way, the geometric fluxes obtained by compactifying M-theory on manifolds with $SU(3)$ structure are more fundamental than the fluxes introduced in (4.6). Since the fluxes in (4.11) and the corresponding F-theory deformations, are simply mapped to the twists (4.8) on the heterotic side, it means that the effect of some of the fluxes in (4.6) can be obtained on the heterotic side by simply considering manifolds with $SU(2)$ structure and shifting the vev of certain scalars. Thus, this resolves a part of the puzzle encountered at the end of section 4.1.

More generally the fluxes (4.6) and the twisting (4.11) are not compatible. To see this note that acting with the exterior derivative on ω_i twice we need that

$$p_{iA}M_B{}^A = 0, \quad \text{and} \quad q_i^A M_A{}^B = 0. \quad (4.17)$$

This means that the twist matrix M must have zero eigenvalues or in other words it means that some of the three-forms or some combination thereof do not change as we go around the circle. These forms will be precisely the ones which are allowed to appear on the right hand side of (4.6).

The same exclusion between fluxes can be seen also on the heterotic side. Here the gauge field fluxes on $K3$ are in general incompatible with the twisting discussed in this section. The reason is the Bianchi identities the field strengths must satisfy⁷

$$dF^I = 0. \quad (4.18)$$

If we try to turn on both the fluxes (4.1) and the twisting, the above Bianchi identity will imply

$$m_A^I T_{\alpha B}{}^A = 0 \quad (4.19)$$

which is precisely of the same form as in type IIA case and tells us that gauge field fluxes can only be turned on along eigenvectors of T_i corresponding to zero eigenvalue.

⁷Note that we are dealing with fluxes for Abelian gauge fields

On the other hand, fluxes which gauge isometries in the vector multiplet sector and those gauging isometries in the hyper-multiplet sector can coexist and their effects can be simply added up.

4.4 Leftover fluxes

Until now we have only discussed the fluxes which have a dual interpretation. It is also important to review which are the fluxes for which the dual is not known. As long as we are talking about vector multiplet gaugings we have seen in section 3 that for all the fluxes there is at least a proposal for their dual. For the hyper-multiplets gaugings the situation is not so simple and there are several fluxes for which a dual is not known. Recall that at the beginning of section 4 we set the parameters m^i in (4.3) to zero. These fluxes introduce magnetic gaugings and it is not clear how to obtain something similar on the heterotic side. The same applies to fluxes which come from compactifying type IIA on manifolds with $SU(3) \times SU(3)$ structure. These fluxes introduce charges which are magnetic dual to the ones in (4.7) and do not have a known heterotic dual. Finally we have already explained that only half of the flux parameters in (4.7) have dual heterotic interpretation.

On the heterotic side we can use manifolds with $SU(2)$ structure which do not have an integrable product structure and the corresponding fluxes do not seem to have a type IIA dual. However it is not clear whether such manifolds with $SU(2)$ structure are meaningful from the point of view of the heterotic type IIA duality.

5 Conclusions

In this paper we have studied the fluxes which can be turned on in heterotic and type IIA compactifications to four dimensions with $N = 2$ supersymmetry from the point of view of the heterotic type IIA duality.

We distinguished two classes of fluxes: fluxes which gauge isometries in the vector multiplet sector and fluxes which gauge isometries in the hyper-multiplet sector. In section 3 we extended the results in [29] and showed that all the fluxes which appear in M-theory compactifications on manifolds with $SU(3)$ structure and which gauge isometries in the vector multiplet sector have a correspondent in heterotic compactifications with duality twists as discussed in [14, 30]. Such fluxes include among others non-geometric fluxes which can be intuitively understood as fluxes for the T-dual gauge fields and have a purely geometric origin on the M-theory side. Furthermore we conjectured that the heterotic R-fluxes introduced in [30] find a geometric realisation on the other side of the duality in the framework of F-theory compactifications on eight-dimensional manifolds with $SU(3)$ structure. For this conjecture we have only presented a few indications including the counting of flux degrees of freedom. A more detailed analysis is needed especially on the F-theory side in order to be sure that the proposed scenario is indeed the correct one. However F-theory compactifications highly depend on the Calabi–Yau manifold used and a general analysis is not possible. One may still use a dual picture,⁸ like M-theory compactified to three-dimensions,⁹ but this is beyond the scope of the present paper.

⁸We thank Eran Palti for pointing this out.

⁹See [41] for recent work on this topic.

For the hyper-multiplet sector the duality map is not very well specified and therefore there is a certain ambiguity in finding the dual fluxes. Still we have been able to identify large classes of fluxes which can be mapped from the heterotic to type IIA side. Again, like in the vector multiplet sector, certain fluxes seem to correspond to M and even F-theory compactifications on manifolds with $SU(3)$ structure.

Dualities in string compactifications with fluxes have played an important role in understanding various fluxes which can be turned on in different situations. We have also seen it in the present paper that by duality we are lead to consider new fluxes in the same spirit as [26]. From this point of view it would be interesting to find the most general setup which is fully invariant under this duality.

Finally we want to comment on the practical use of the results in this paper. Since we are dealing with $N = 2$ supergravities the possibility to apply these results to phenomenology is quite remote. We nevertheless want to point out that even the $N = 2$ analysis may be useful at least for deriving the low energy effective actions in $N = 1$ compactifications which use similar backgrounds. Also one may consider various projections/truncations similar to the orientifolding in type II compactifications such that the final theory has only $N = 1$ supersymmetry. Last but not least, one may check if the theories described here exhibit spontaneous $N = 2 \rightarrow N = 1$ breaking [42]. From this point of view, the only setup which may be suitable for such an analysis is heterotic compactifications with R -fluxes as described in section 3. Indeed as explained there, heterotic compactification naturally take us into a symplectic frame where no prepotential exists and moreover rotations to a basis where a prepotential exists induce magnetic gaugings along with the existing electric ones which is a necessary condition for a $N = 2$ supergravity to present spontaneous $N = 2 \rightarrow N = 1$ breaking.

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Appendix

A $N = 2$ (gauged) supergravity in four dimensions

In this appendix we shall review the main features of $N = 2$ supergravity in four dimensions. As the compactifications we are dealing with fall in this class the formulae here will be applicable to both type IIA and heterotic pictures. We shall only be concerned with the bosonic fields and therefore we shall largely ignore the fermions whose interactions can be obtained by supersymmetry.

The $N = 2$ supergravity multiplet contains the graviton $g_{\mu\nu}$ and a vector field, the graviphoton. Other $N = 2$ multiplets which will be of interest for us are the vector multiplets and the hyper-multiplets. The vector multiplets contain one vector field and one complex scalar in the adjoint of the gauge group. The hyper-multiplets contain four scalar

fields and are responsible for the matter content of the theory. $N = 2$ supersymmetry requires that the manifold spanned by the scalars splits into a product of special Kähler manifold – which describes the scalars in the vector multiplets – and a quaternionic manifold which describes the hyper-scalars

$$\mathcal{M}_{scalar} = \mathcal{M}_{SK} \times \mathcal{M}_Q . \quad (\text{A.1})$$

For the issues discussed in this paper, the quaternionic manifold does not play any special role. In type IIA compactifications the quaternionic metric can be written explicitly in terms of quantities defined on the Calabi–Yau manifold, while in the heterotic case the metric is not known in general.

On the special-Kähler manifold \mathcal{M}_{SK} we can introduce projective coordinates, $X^I, I = 0, \dots, n_v$, in terms of which the scalars in the vector multiplet sector are given by

$$x^i = \frac{X^i}{X^0} , \quad i = 1, \dots, n_v , \quad (\text{A.2})$$

where X^0 is supposed to be non-vanishing. The geometry is then described entirely by a holomorphic function, called prepotential, $\mathcal{F}(X^I)$, which is homogeneous of degree two in the projective variables X^I . The Kähler potential is given in terms of the prepotential as

$$K = -\ln(X^I \bar{\mathcal{F}}_I - \bar{X}^I \mathcal{F}_I) , \quad (\text{A.3})$$

where $\mathcal{F}_I = \partial_{X^I} \mathcal{F}$ denote the derivatives of the prepotential. Moreover, the same function \mathcal{F} gives the couplings of the gauge fields in the vector multiplets

$$\mathcal{N}_{IJ} = \bar{\mathcal{F}}_{IJ} + 2i \frac{\text{Im} \mathcal{F}_{IK} \text{Im} \mathcal{F}_{JL} X^K X^L}{\text{Im} \mathcal{F}_{KL} X^K X^L} \quad (\text{A.4})$$

where $\mathcal{F}_{IJ} = \partial_{X^I} \partial_{X^J} \mathcal{F}$. The imaginary part of the above matrix describes the generalised coupling constants while the real part the generalised theta angles. Altogether the bosonic part of the $N = 2$ supergravity action is given by

$$S = \int \left[\frac{1}{2} R * \mathbf{1} - g_{ij} dx^i \wedge * d\bar{x}^j - h_{uv} dq^u \wedge * dq^v + \frac{1}{4} \text{Im} \mathcal{N}_{IJ} F^I \wedge * F^J + \frac{1}{4} \text{Re} \mathcal{N}_{IJ} F^I \wedge F^J \right] . \quad (\text{A.5})$$

Any (global) symmetry of this action must necessarily be an isometry of the scalar manifold (A.1). Some of these symmetries can be made local (gauged) and in this case the partial derivatives in the kinetic terms for the scalars are replaced by appropriate covariant derivatives

$$dx^i \rightarrow Dx^i = dx^i - k_I^i A^I ; \quad dq^u \rightarrow Dq^u = dq^u - k_I^u A^I , \quad (\text{A.6})$$

where k_I^u and k_I^i are the components of the Killing vectors which give the directions in the scalar space which are gauged. The holomorphic Killing vectors k_I^i can be obtained as derivatives of a holomorphic prepotential \mathcal{P}_I , while k_I^u can be obtained as covariant derivatives (on the quaternionic space of hyper-multiplets) of a triplet of prepotentials \mathcal{P}_I^x . Since they do not play any role in the paper we shall not discuss them in the following and refer the interested reader to the existing literature [33]. Finally, $N = 2$ supersymmetry requires the presence of a scalar potential in connection with the gaugings above, which in terms of the prepotentials defined above is given by

$$V = e^K X^I \bar{X}^J (g_{\bar{i}j} k_I^{\bar{i}} k_J^j + 4h_{uv} k_I^u k_J^v) - \left[\frac{1}{2} (\text{Im} \mathcal{N})^{-1 IJ} + 4e^K X^I X^J \right] P_I^x P_J^x . \quad (\text{A.7})$$

In the case when the gauge group is non-Abelian, the fields X^I are charged as they transform in the adjoint representation of the gauge group along with the vector fields. If the prepotential defining the special Kähler geometry, \mathcal{F} , is invariant under the gauge transformations, then the $N = 2$ action is obviously invariant under gauge transformations. However it is also possible that the prepotential is not invariant and one can still define an invariant action. This is the case when the prepotential changes under gauge transformations by a term

$$\delta\mathcal{F} = \Lambda^I C_{IJK} X^J X^K , \quad (\text{A.8})$$

where Λ^I denote the gauge transformation parameters and C_{IJK} are real constants. For the transformation above, the Kähler potential changes by the real part of a holomorphic function (thus leaving the Kähler metric invariant) while kinetic terms for the gauge fields are left invariant. However, the generalised theta terms do change and in order to reestablish the gauge invariance of the action one has to add the following term

$$\frac{1}{3} \int C_{IJK} A^I \wedge A^J \wedge \left(dA^K - \frac{3}{8} f_{LM}^K A^L \wedge A^M \right) . \quad (\text{A.9})$$

In the end let us discuss the electric magnetic duality of $N = 2$ supergravities which is of central importance to the work presented here. This duality does not represent an invariance of the action but rather a symmetry of the equations of motion together with the Bianchi identities. Let us define the magnetic field strengths as

$$G_I = \frac{\partial \mathcal{L}}{\partial F^I} , \quad (\text{A.10})$$

where \mathcal{L} denotes the Lagrange density of the $N = 2$ supergravity theory. The system of equations of motion and Bianchi identities (in the ungauged theory) which read

$$dG_I = 0 , \quad dF^I = 0 , \quad (\text{A.11})$$

is invariant under symplectic rotations

$$\begin{pmatrix} F^I \\ G_I \end{pmatrix} \rightarrow \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} F^I \\ G_I \end{pmatrix} . \quad (\text{A.12})$$

where U, V, W and Z are constant, real matrices which obey

$$\begin{aligned} U^T V - W^T Z &= V^T U - Z^T W = \mathbf{1} , \\ U^T W &= W^T U , \quad Z^T V = V^T Z . \end{aligned} \quad (\text{A.13})$$

Under this transformation (X^I, \mathcal{F}_I) form a symplectic vector which transforms precisely as the vector (F^I, G_I) . Finally, the gauge coupling matrix \mathcal{N} transforms as

$$\mathcal{N} \rightarrow (V\mathcal{N} + W)(U + Z\mathcal{N})^{-1} \quad (\text{A.14})$$

It can be easily seen that in general there exist symplectic transformations such that in the resulting frame \mathcal{F}_I are no longer the derivatives of one function \mathcal{F} and therefore in such frames, the prepotential does not exist. Nevertheless, the vector (X^I, \mathcal{F}_I) is enough for defining the geometry and the Kähler potential is again given by the formula (A.3).

Compactifications of the heterotic string on $K3 \times T^2$ manifolds naturally leads to special geometries in a frame where no prepotential exists.

In the gauged theory as presented above, the symplectic symmetry is broken by choosing the “electric” vector fields which participate in the gauging.¹⁰ The fact that in the heterotic case one ends up in a symplectic frame where no prepotential exists is crucial for explaining the duality with type IIA compactifications.

B The vector multiplet sector in the heterotic compactifications

Much of the structure of the vector multiplet sector in heterotic compactifications on $K3 \times T^2$ comes from the compactification on T^2 . In general T^d compactifications have 16 supercharges (hence $N = 4$ in four dimensions) and therefore the vector multiplet sector inherits much of the structure of $N = 4$ theories. In particular the torus moduli together with the Wilson lines parameterise a $O(d, d + 16)$ matrix as follows [38]

$$\mathcal{M}_{IJ} = \begin{pmatrix} g + C^T g^{-1} C + A A^T & -C^T g^{-1} & C^T g^{-1} A + A \\ -g^{-1} C & g^{-1} & -g^{-1} A \\ A^T g^{-1} C + A & -A^T g^{-1} & \mathbf{1}_{16 \times 16} + A^T g^{-1} A \end{pmatrix} \quad (\text{B.1})$$

In the above we have used matrix notation (with matrix multiplication assumed) and the various quantities have the following meaning: g is a $d \times d$ matrix representing the metric on the d -dimensional torus, A is a $d \times 16$ matrix made of the gauge fields A^a , $a = 1, \dots, 16$ with legs on T^d and C is a $d \times d$ matrix which is given by $C = B + \frac{1}{2} A A^T$, where B denotes the B -field on T^d .

We are obviously interested in the case $d = 2$ which means that the matrix above is a 20×20 matrix. We also have to keep in mind that the $K3$ part of the compactification may influence the above by the fact that the gauge bundle which we need to turn on, may break some of the original gauge symmetry and therefore the dimension of the scalar space may not be 36 as it should have been in this case, but can be smaller.¹¹ Consequently the matrix \mathcal{M} above may be smaller and we shall work with a moduli matrix parameterising a $O(2, n_v - 1)$ group element. As in the $N = 4$ theory, the kinetic term for the moduli is given as the standard kinetic term on a group, namely $\text{tr}(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M})$. In the $N = 2$ theory, the scalar manifold spanned by the scalars in the vector multiplets is a special Kähler manifold and it is meaningful to write the kinetic term in the appropriate way.

¹⁰There exist a formalism – aka the embedding tensor formalism [43]– in which the $N = 2$ gauged supergravity is written in an explicit symplectic covariant way. For this one introduces magnetic gaugings and allows the electric and magnetic charges to be rotated into one another by the symplectic rotations.

¹¹In principle, some of the KK gauge fields which appear in the T^2 compactification may be broken by the gauge bundle on $K3$. We shall however consider these gauge fields survive the compactification as they constitute the most interesting sector when fluxes are turned on.

This is done by defining complex coordinates on the space as

$$\begin{aligned}
A_9^a &= \sqrt{2} \frac{n^a - \bar{n}^a}{u - \bar{u}} ; & A_8^a &= \sqrt{2} \frac{\bar{u}n^a - u\bar{n}^a}{u - \bar{u}} \\
B_{98} &= \frac{1}{2} \left[(t + \bar{t}) - \frac{(n^a + \bar{n}^a)(n^a - \bar{n}^a)}{u - \bar{u}} \right] \\
\sqrt{g} &= -\frac{i}{2} \left[(t - \bar{t}) - \frac{(n^a - \bar{n}^a)(n^a - \bar{n}^a)}{u - \bar{u}} \right] \\
g_{99} &= \frac{2i}{u - \bar{u}} \sqrt{g} ; & g_{89} &= i \frac{u + \bar{u}}{u - \bar{u}} \sqrt{g}
\end{aligned} \tag{B.2}$$

Finally there is the dialton field, ϕ , together with its partner, the axion, a , which is Poincare dual to the B -field in four dimensions which combine as

$$s = a - ie^\phi . \tag{B.3}$$

The special geometry data is now given as

$$\begin{aligned}
X^0 &= 1 ; & X^1 &= ut - n^a n^a ; & X^2 &= -u ; & X^3 &= t ; & X^a &= \sqrt{2} n^a ; \\
\mathcal{F}_0 &= s(ut - n^2) ; & \mathcal{F}_1 &= s ; & \mathcal{F}_2 &= st ; & \mathcal{F}_3 &= -su ; & \mathcal{F}_a &= \sqrt{2} s n^a ;
\end{aligned} \tag{B.4}$$

Note that the symplectic frame above is such that \mathcal{F}_I are not derivatives of a prepotential and therefore in this basis no prepotential exists. By a symplectic transformation ($X^1 \rightarrow \tilde{X}^1 = \mathcal{F}_1$ and $\mathcal{F}_1 \rightarrow \tilde{\mathcal{F}}_1 = -X^1$) we reach a basis where a prepotential exists. This is precisely the type IIA basis and the prepotential is given by (2.11) with intersection numbers (3.8).

To end this section, note that because in the natural heterotic symplectic basis there is no prepotential, the gauge coupling matrix can not be computed as in (A.4). One can use the more complicated formalism of [33] or use a detour to go first to a symplectic frame where a prepotential exists, compute the gauge coupling matrix there and then perform the inverse rotation to the original symplectic frame and use formulae (A.14) to transform the gauge coupling matrix. One can also obtain the gauge coupling matrix directly from the compactification and the result is given by

$$Im\mathcal{N}_{IJ} = \frac{s - \bar{s}}{2i} \mathcal{M}_{IJ} , \quad Re\mathcal{N}_{IJ} = -\frac{s + \bar{s}}{2} \eta_{IJ} , \tag{B.5}$$

for the matrix \mathcal{M}_{IJ} defined in (B.1) for the special case of T^2 compactifications.

B.1 Gaugings in the vector multiplet sector

The purpose of this appendix is to show the calculations which lead to the gaugings (3.33) and implicitly to (3.30) in the particular case when there are no R-fluxes. The calculation is straightforward and one starts from the general form for the covariant derivative (3.29) and substitutes the matrices (3.28) and (3.24) using the definitions for the fields (B.2). In order to ease the calculation it is worth noting that the scalars V_8^M have a simple form when expressed in terms of the fields t, u and n^a . In particular we find

$$V_8^3 = \frac{u + \bar{u}}{2} ; \quad V_8^4 = -\frac{t + \bar{t}}{2} ; \quad V_8^a = -\frac{n^a + \bar{n}^a}{\sqrt{2}} . \tag{B.6}$$

Let us now expand formula (3.29) using the matrices (3.28) and (3.24) with the structure constants (3.19) and (3.32). From (B.6) we see that the covariant derivative of the element $\tilde{\mathcal{M}}_{2M}$ will give us valuable information about the covariant derivatives of the fields u , t and n^a . Actually we shall argue that using properties like the holomorphy of Killing vectors, the evaluation of this covariant derivative will be enough to determine without any ambiguity the form of the covariant derivatives of the fields u , t and n^a . In order to evaluate the covariant derivative of the matrix element $\tilde{\mathcal{M}}_{2N}$ we first need to write the one of the element $\tilde{\mathcal{M}}_{22} = g_{88}^{-1} \equiv g^{88}$. We find

$$D_\mu g^{88} = \partial_\mu g^{88} - 2g^{88} \tilde{N}_{NP} V_8^P A_\mu^N. \quad (\text{B.7})$$

Note that this covariant derivative is non-trivial only if R-fluxes are present.

Before we compute the covariant derivative of the element $\tilde{\mathcal{M}}_{2N}$ we should clarify one point about the notations we use. The capital indices I, J , etc. in formula (3.29) run over all the vector fields in the theory while the indices N, P , etc. run over two less vector fields (ie the ones which appear in the last step of the compactification from five to four dimensions). We have therefore used the notation $I = (1, 2, N)$. Therefore the indices N, P , etc. are understood to range from $3, \dots, n_v$. As an example, the first element of the matrix \mathcal{M}_{NP} – ie the element $(1, 1)$ in standard notation – will be denoted by \mathcal{M}_{33} .

With the above observations, from the covariant derivative of the matrix element $\tilde{\mathcal{M}}_{1N}$ we find

$$\begin{aligned} D_\mu (L_{NP} V_8^P) &= - (D_\mu g_{88}) \tilde{\mathcal{M}}_{2N} - g_{88} (D_\mu \tilde{\mathcal{M}}_{2N}) \\ &= \partial_\mu (L_{NP} V_8^P) - N_{NQ} (V_8^Q A_\mu^1 + A_\mu^Q) - \tilde{N}_{NQ} V_8^Q A_\mu^2 \\ &\quad + \left(\tilde{N}_{QR} V_8^R L_{NP} V_8^P - g_{88} \tilde{N}_Q^R \mathcal{M}_{RN} + \frac{1}{2} \tilde{N}_{NQ} L_{PR} V_8^P V_8^R \right) A_\mu^Q \end{aligned} \quad (\text{B.8})$$

We can now specialise for various values of the index $N = 3, 4, \dots, n_v + 1$. Using the definitions (B.2) in order to express the elements of the matrix \mathcal{M} and the parameterisation of the twist matrices (3.18) and (3.31) we find

$$\begin{aligned} D_\mu (u + \bar{u}) &= \partial_\mu (u + \bar{u}) - \tilde{N}_{J4} (n^a n^a + \bar{n}^a \bar{n}^a) A_\mu^J + \tilde{N}_{J3} (u^2 + \bar{u}^2) A_\mu^J \\ &\quad - \sqrt{2} \tilde{N}_{Ja} (n^a u + \bar{n}^a \bar{u}) A_\mu^J - 2N_{4J} (A_\mu^J + V_8^J A_\mu^1) - 2\tilde{N}_{4J} V_8^J A_\mu^2 \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} D_\mu (t + \bar{t}) &= \partial_\mu (t + \bar{t}) - \tilde{N}_{1J} (n^a n^a + \bar{n}^a \bar{n}^a) A_\mu^J - \tilde{N}_{J4} (t^2 + \bar{t}^2) A_\mu^J \\ &\quad - \sqrt{2} \tilde{N}_{Ja} (n^a t + \bar{n}^a \bar{t}) A_\mu^J + 2N_{3J} (A_\mu^J + V_8^J A_\mu^1) + 2\tilde{N}_{3J} V_8^J A_\mu^2 \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned} D_\mu (n^a + \bar{n}^a) &= \partial_\mu (n^a + \bar{n}^a) + \tilde{N}_{J3} (n^a u + \bar{n}^a \bar{u}) A_\mu^J - \tilde{N}_{J4} (n^a t + \bar{n}^a \bar{t}) A_\mu^J \\ &\quad - \sqrt{2} \tilde{N}_{Jb} (n^b n^a + \bar{n}^b \bar{n}^a) A_\mu^J - \frac{1}{\sqrt{2}} \tilde{N}_{Ja} (u t + \bar{u} \bar{t} - n^b n^b - \bar{n}^b \bar{n}^b) A_\mu^J \\ &\quad + \frac{1}{\sqrt{2}} \tilde{N}_{aJ} V_8^J A_\mu^2 + \frac{1}{\sqrt{2}} N_{aJ} (A_\mu^J + V_8^J A_\mu^1) \end{aligned} \quad (\text{B.11})$$

Similarly, using other elements of the matrix $\tilde{\mathcal{M}}$ one can derive the covariant derivatives of the imaginary parts of the fields. However, using the holomorphy of the Killing vectors we can already read off from the expressions above what the covariant derivatives of the complex fields u , t and n^a are. The only ambiguity can come from the constant terms in Killing vectors. However, such constant terms can not appear in the covariant derivatives

of the imaginary parts of the fields u , t , and n^a , as such terms correspond to gaugings of shift isometries and the imaginary parts of the fields do not have such invariances in the ungauged theory.¹² With this one immediately sees that the relations above imply the covariant derivatives written in (3.33). As a consistency check, one can verify, after a lengthy, but completely straightforward calculation, that the Killing vectors which can be read from these equations satisfy the commutation relations

$$[k_I, k_J] = f_{IJ}^K k_K, \quad (\text{B.12})$$

with the structure constants f_{IJ}^K defined in (3.19) and (3.32).

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¹²Note that the imaginary parts of the fields appear explicitly in the Kähler potential and therefore the theory does not have shift symmetries in these directions.

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